ON THE GLOBAL LOG CANONICAL THRESHOLD OF FANO COMPLETE INTERSECTIONS

THOMAS ECKL AND ALEKSANDR PUKHLIKOV

Abstract. We show that the global log canonical threshold of generic Fano complete intersections of index 1 and codimension k in $\mathbb{P}^{M+k}$ is equal to 1 if $M \geqslant 3k+4$ and the highest degree of defining equations is at least 8. This improves the earlier result where the inequality $M \geqslant 4k+1$ was required, so the class of Fano complete intersections covered by our theorem is considerably larger. The theorem implies, in particular, that the Fano complete intersections satisfying our assumptions admit a Kähler-Einstein metric. We also show the existence of Kähler-Einstein metrics for a new finite set of families of Fano complete intersections.
Bibliography: 10 titles.

1. The canonical and log canonical thresholds. Consider a smooth Fano variety X, such that $\mathrm{Pic}\, X = \mathbb{Z}K_X$, $D \sim -nK_X$ an effective divisor, $n \geq 1$.

Definition 1. The pair $(X, \alpha D)$, where $\alpha \in \mathbb{Q}$, is *not canonical* (respectively, *not log canonical*), if there exists a prime divisor E over X such that the inequality

$$\alpha \,\mathrm{ord}_E\, D > a(E)$$

(respectively, $\alpha \,\mathrm{ord}_E\, D > a(E) + 1$) is satisfied, where $a(E) = a(X, E)$ is the discrepancy of E with respect to the model X.

Explicitly, this means that there are a birational morphism $\varphi\colon \widetilde{X} \to X$, where $\widetilde{X}$ is a smooth projective variety, and a prime φ-exceptional divisor $E \subset \widetilde{X}$ such that

$$\alpha \,\mathrm{ord}_E\, \varphi^* D > a(E)$$

(respectively, $\alpha \,\mathrm{ord}_E\, \varphi^* D > a(E) + 1$).

Definition 2. The global canonical (respectively, log canonical) threshold of the variety X is defined by the equality

$$\mathrm{ct}(X) = \sup\{\lambda \in \mathbb{Q}_+ \,|\, \text{the pair } (X, \frac{\lambda}{n} D) \text{ is canonical for all } D \in |-nK_X|\},$$

(respectively,

$$\mathrm{lct}(X) = \sup\{\lambda \in \mathbb{Q}_+ \,|\, \text{the pair } (X, \frac{\lambda}{n} D) \text{ is log canonical for all } D \in |-nK_X|\}).$$

(Note that as $D \sim -nK_X$, the integer $n \geqslant 1$ depends on the effective divisor D.)

The importance of canonical and log canonical thresholds comes from their applications to complex differential geometry and to certain problems of higher-dimensional birational geometry.

1991 *Mathematics Subject Classification.* 14E05, 14E07, 14J45.

Key words and phrases. Fano variety, birational rigidity, Kähler-Einstein metric, hypertangent divisor.

In [2, 3, 10] the following fact was shown (see [1] for a detailed discussion and precise references).

Theorem 1. *Assume that the inequality*

$$\operatorname{lct}(X) > \frac{\dim X}{\dim X + 1}$$

holds. Then on the variety X there exists a Kähler-Einstein metric.

The log canonical threshold is important in this differential-geometric context because it indicates the non-triviality of certain multiplier ideal sheaves. In their analytic interpretation, these multiplier ideal sheaves in turn measure the failure of a priori estimates sufficient to solve a Monge-Ampère equation for a Kähler-Einstein metric.

Definition 3. (i) The *mobile canonical threshold* of a variety X, which is denoted by the symbol $\operatorname{mct}(X)$, is the supremum of such $\lambda \in \mathbb{Q}_+$ that the pair $(X, \frac{\lambda}{n} D)$ is canonical for a generic divisor $D \in \Sigma$ of any mobile linear system $\Sigma \subset |-nK_X|$ (that is to say, any system Σ with no fixed components).

(ii) The *threshold of canonical adjunction* $c(D, X)$, where D is an effective divisor on the variety X, is the supremum of such $\lambda \in \mathbb{Q}_+$ that the $\mathbb{Q}$-divisor $D + \lambda K_X$ is pseudoeffective. For a mobile linear system Σ on X its *virtual threshold of canonical adjunction* $c_{\mathrm{virt}}(\Sigma, X)$ is $\inf\{c(\widetilde{D}, \widetilde{X})\}$, where the infimum is taken over all models $\widetilde{X} \to X$ and $\widetilde{D}$ is the strict transform of a generic divisor $D \in \Sigma$.

(iii) The variety X is said to be *birationally superrigid* if $c(\Sigma, X) = c_{\mathrm{virt}}(\Sigma, X)$ for any mobile linear system on X.

The property of being birationally superrigid gives an exhaustive description of the birational geometry of the given variety, as can be seen, for instance, in the claims (i)-(v) of the following theorem, shown in [6]: they are all easy implications of birational superrigidity.

Theorem 2. *Assume that primitive Fano varieties $F_1, \dots, F_K$, $K \geq 2$, satisfy the conditions* $\operatorname{lct}(F_i) = \operatorname{mct}(F_i) = 1$. *Then their direct product*

$$V = F_1 \times \cdots \times F_K$$

is a birationally superrigid variety. In particular,

(i) *Every structure of a rationally connected fiber space on the variety V is given by a projection onto a direct factor. More precisely, let $\beta\colon V^\sharp \to S^\sharp$ be a rationally connected fiber space and $\chi\colon V \dashrightarrow V^\sharp$ a birational map. Then there exists a subset of indices*

$$I = \{i_1, \dots, i_k\} \subset \{1, \dots, K\}$$

and a birational map

$$\alpha\colon F_I = \prod_{i \in I} F_i \dashrightarrow S^\sharp,$$

such that the diagram

$$\begin{array}{ccccc} & V & \overset{\chi}{\dashrightarrow} & V^\sharp & \\ \pi_I & \downarrow & & \downarrow & \beta \\ & F_I & \overset{\alpha}{\dashrightarrow} & S^\sharp & \end{array}$$

commutes, that is, $\beta \circ \chi = \alpha \circ \pi_I$*, where*

$$\pi_I \colon \prod_{i=1}^{K} F_i \to \prod_{i \in I} F_i$$

is the natural projection onto a direct factor.

(ii) *Let* $V^\sharp$ *be a variety with* $\mathbb{Q}$*-factorial terminal singularities, satisfying the condition*

$$\dim_{\mathbb{Q}}(\operatorname{Pic} V^\sharp \otimes \mathbb{Q}) \leqslant K,$$

and $\chi \colon V \dashrightarrow V^\sharp$ *a birational map. Then* χ *is a (biregular) isomorphism.*

(iii) *The groups of birational and biregular self-maps of the variety* V *coincide:*

$$\operatorname{Bir} V = \operatorname{Aut} V.$$

In particular, the group $\operatorname{Bir} V$ *is finite.*

(iv) *The variety* V *admits no structures of a fibration into rationally connected varieties of dimension strictly smaller than* $\min\{\dim F_i\}$*. In particular,* V *admits no structures of a conic bundle or a fibration into rational surfaces.*

(v) *The variety* V *is non-rational.*

For the details about birational (super)rigidity, a discussion of its properties and examples of birationally (super)rigid varieties, see [9].

2. Fano complete intersections. Fix an integer $k \geqslant 2$. Consider an arbitrary system $(d_1, \dots, d_k)$ of positive integers, satisfying the condition $d_k \geqslant \dots \geqslant d_1 \geqslant 2$. Set $M = d_1 + \dots + d_k - k$. Fix the complex projective space $\mathbb{P} = \mathbb{P}^{M+k}$ and consider the family $\mathcal{F}(d_1, \dots, d_k)$ of non-singular complete intersections V of the type $d_1 \cdot \dots \cdot d_k$ in $\mathbb{P}$:

$$V = F_1 \cap \dots \cap F_k \subset \mathbb{P},$$

Here $F_i \subset \mathbb{P}$ is a hypersurface of degree d_i, and $\operatorname{codim} V = k$.

The following two theorems collect the known information about the global canonical and log canonical thresholds of Fano complete intersections as above.

Theorem 3. *Assume that* $M \geqslant 4k+1$ *and* $d_k \geqslant 8$*. Then for a generic (in the sense of Zariski topology on the space* $\mathcal{F}(d_1, \dots, d_k)$*) variety* $V \in \mathcal{F}(d_1, \dots, d_k)$ *the equality* $\operatorname{ct}(V) = 1$ *holds.*

Proof: see [7, Section 3].

Thus under the assumptions of Theorem 3 on V exists a Kähler-Einstein metric. Besides, since $\operatorname{lct}(V) \geqslant \operatorname{ct}(V)$ and $\operatorname{mct}(V) \geqslant \operatorname{ct}(V)$, the variety V satisfies the assumptions of Theorem 2 and for that reason can be used as a direct factor in birationally superrigid Fano direct products.

Theorem 4. *Assume that* $M \geqslant 4k+1$ *and any of the following conditions holds:*
(i) $d_k = d_{k-1} = 7$ *and* $M \leq 47$,
(ii) $d_k = 7$, $d_{k-1} \leq 6$ *and* $M \leq 19$.
(iii) $k = 2$, $d_1 = d_2 = 6$, $M = 10$.

Then the canonical threshold $\mathrm{ct}(V)$ *(and hence also the log canonical threshold* $\mathrm{lct}(V)$*) of a generic variety* $V \in \mathcal{F}(d_1, \ldots, d_k)$ *satisfies the inequality*

$$\mathrm{ct}(V) > \frac{M}{M+1}.$$

Proof: see [8].

Therefore, on a general variety $V \in \mathcal{F}(d_1, \ldots, d_k)$, satisfying one of the conditions listed in Theorem 4 there exists a Kähler-Einstein metric.

The aim of the present note is to improve the claims of Theorems 3 and 4, extending them to a larger class of Fano complete intersections of index 1. We will show the following two facts.

Theorem 5. *Assume that* $M \geqslant 3k+4$ *and* $d_k \geqslant 8$. *Then for a generic (in the sense of Zariski topology on the space* $\mathcal{F}(d_1, \ldots, d_k)$*) variety* $V \in \mathcal{F}(d_1, \ldots, d_k)$ *the equality* $\mathrm{ct}(V) = 1$ *holds.*

Theorem 6. *Assume that* $M \geqslant 3k+4$ *and any of the two following conditions holds:*

(i) $d_k = d_{k-1} = 7$ *and* $M \leqslant 47$,

(ii) $d_k = 7$, $d_{k-1} \leqslant 6$ *and* $M \leqslant 19$.

(iii) $k = 2$, $d_1 = d_2 = 6$, $M = 10$.

Then the canonical threshold $\mathrm{ct}(V)$ *(and hence also the log canonical threshold* $\mathrm{lct}(V)$*) of a generic variety* $V \in \mathcal{F}(d_1, \ldots, d_k)$ *satisfies the inequality*

$$\mathrm{ct}(V) > \frac{M}{M+1}.$$

Remark 1. Theorem 5 covers a considerably larger class of Fano complete intersections than Theorem 3. The same is true for the part (i) of Theorems 6 and 4. For the part (iii), Theorem 6 gives nothing new compared to Theorem 4. As for the part (ii), Theorem 6 gives the existence of the Kähler-Einstein metric for the following 7 families of Fano complete intersections that do not fit into the assumptions of Theorem 4, all of them in $\mathbb{P}^{24}$:

$$\mathcal{F}(2,5,5,5,7), \quad \mathcal{F}(2,4,5,6,7), \quad \mathcal{F}(2,3,6,6,7), \quad \mathcal{F}(3,3,5,6,7),$$
$$\mathcal{F}(3,4,5,5,7), \quad \mathcal{F}(3,4,4,6,7), \quad \mathcal{F}(4,4,4,5,7).$$

3. The conditions of general position. Now we will explain what we mean by the genericity of a Fano complete intersection $V \in \mathcal{F}(d_1, \ldots, d_k)$. Fix any point $o \in V$ and let $(z_1, \ldots, z_{M+k})$ be a system of affine coordinates on $\mathbb{P}$ with the origin at the point o,

$$f_i = q_{i,1} + \cdots + q_{i,d_i}$$

the equation of the hypersurface F_i with respect to (non-homogeneous) coordinates z_*, decomposed into homogeneous in z_* components $q_{i,j}$, $\deg q_{i,j} = j$. Since V is a non-singular variety, the system of linear equations

$$q_{1,1} = \cdots = q_{k,1} = 0$$

defines a linear subspace $T_oV \subset \mathbb{C}^{M+k}$ of codimension k, the tangent space to the variety V at the point o. We define a finite set of pairs $I \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ in the following way: if $d_{k-1} = d_k$, then we set

$$I = \{(i,j) \,|\, 1 \leqslant i \leq k, 1 \leqslant j \leqslant d_i, (i,j) \notin \{(k,d_k),(k-1,d_{k-1})\}\},$$

and if $d_{k-1} \leq d_k - 1$, then we set

$$I = \{(i,j) \,|\, 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant d_i, (i,j) \notin \{(k,d_k),(k,d_k-1)\}\}.$$

Definition 4. We say that the complete intersection V is *regular at the point o*, if for any linear form $l(z_*)$, not vanishing identically on the subspace T_oV, the set of homogeneous polynomials

$$\{l\} \cup \{q_{i,j} \,|\, (i,j) \in I\}$$

forms a regular sequence in $\mathcal{O}_{o,\mathbb{P}}$, that is, the system of equations in $\mathbb{C}^{M+k}$

$$l = 0, \quad q_{i,j} = 0, (i,j) \in I,$$

defines a subset of codimension $\#I+1$. Finally, we say that the complete intersection V is *regular*, if it is regular at every point.

In Sec. 4 below we show the following fact.

Theorem 7. *For $M \geqslant 3k+4$ there exists a non-empty Zariski open subset*

$$\mathcal{F}_{\mathrm{reg}}(d_1,\dots,d_k) \subset \mathcal{F}(d_1,\dots,d_k),$$

such that any variety $V \in \mathcal{F}_{\mathrm{reg}}(d_1,\dots,d_k)$ is regular.

By genericity in Theorems 3-6 we mean the regularity. For that reason, the main results of this paper (Theorems 5,6) are essentially dependent on Theorem 7 which allows us to assume regularity of the complete intersection V.

4. Proof of the regularity conditions. Let us show Theorem 7. The proof proceeds in a series of reduction steps and case distinctions, through the estimates (1) - (12).

Set $\mathcal{P}(d_1,\dots,d_k) := \prod_{i=1}^k H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(d_k))$, and let the incidence variety $\mathcal{V} \subset H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(1)) \times \mathbb{P}^{M+k} \times \mathcal{P}(d_1,\dots,d_k)$ consist of tuples $(L,o,F_1,\dots,F_k)$ such that $o \in \{L = F_1 = \dots = F_k = 0\}$. Let p,q,r be the natural projections of $\mathcal{V}$ to $H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(1))$, $\mathbb{P}^{M+k}$ resp. $\mathcal{P}(d_1,\dots,d_k)$.

The image of $\mathcal{V}$ under the projection $p \times q$ is the incidence variety

$$\mathcal{L} \subset H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(1)) \times \mathbb{P}^{M+k}$$

consisting of pairs (L,o) such that $L(o) = 0$. All the $(p \times q)$-fibers $\mathcal{V}_{L,o} \subset \mathcal{V}$ over points $(L,o) \in \mathcal{L}$ are isomorphic to a product of affine spaces and have codimension k in $\mathcal{P}(d_1,\dots,d_k)$, as vanishing in o imposes one linear condition on the sections in $H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(d_i))$. Hence $\mathcal{V}$ is irreducible.

Bertini's theorem shows that for a general tuple $(F_1,\dots,F_k) \in \mathcal{P}(d_1,\dots,d_k)$ the algebraic subset $\{F_1 = \dots = F_k = 0\}$ is an M-dimensional complete intersection. Hence the general r-fiber in $\mathcal{V}$ has dimension $(M+k)+M = 2M+k$.

Let $\mathcal{V}^{\mathrm{sm,nonreg}}$ be the locally Zariski-closed subset of tuples $(L,o,F_1,\dots,F_k) \in \mathcal{V}$ such that $V = \{F_1 = \ldots = F_k = 0\}$ is a smooth complete intersection of dimension M, but the dehomogenisation l of the linear form L in affine coordinates around o and the homogeneous parts $q_{i,j}$ of the dehomogenized F_i do not satisfy the regularity condition of Definition 4. Then the theorem is shown if the projection of the Zariski-closure of $\mathcal{V}^{\mathrm{sm,nonreg}}$ does not cover $\mathcal{P}(d_1, \ldots, d_k)$.

To show this claim we note that every r-fiber of $\mathcal{V}^{\mathrm{sm,nonreg}}$ must be at least k-dimensional: If the subscheme $V = \{F_1 = \ldots = F_k = 0\} \subset \mathbb{P}^{M+k}$ is smooth of dimension M in $o \in \mathbb{P}^{M+k}$ then we can choose homogeneous coordinates $[Z_0 : \ldots : Z_{M+k}]$ on $\mathbb{P}^{M+k}$ such that $o = [1 : 0 : \ldots : 0]$ and $q_{1,1} = z_{M+1}, \ldots, q_{k,1} = z_{M+k}$ in the affine coordinates $z_1, \ldots, z_{M+k}$ dehomogenized with respect to Z_0. Hence there is a k-dimensional linear subspace of linear forms $L \in H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(1))$ with the same intersection of $\{L = 0\} \subset \mathbb{P}^{M+k}$ and the tangent space T_oV, seen as a linear subspace of $\mathbb{P}^{M+k}$. In particular, if the sequence $l, q_{1,1}, \ldots, q_{k,1}, q_{1,2}, \ldots$ is not regular for one such linear form L (dehomogenized to l) then the sequence will not be regular for any other linear form in this k-dimensional linear subspace. Consequently, we only need to show that

$$\mathrm{codim}_{\mathcal{V}} \mathcal{V}^{\mathrm{sm,nonreg}} \geq (2M + k) - k + 1 = 2M + 1. \tag{1}$$

The q-fibers $\mathcal{V}_o$ over points $o \in \mathbb{P}^{M+k}$ are all isomorphic. Consequently it is enough to show for all the subsets $\mathcal{V}^{\mathrm{sm,nonreg}} \cap \mathcal{V}_o =: \mathcal{V}_o^{\mathrm{sm,nonreg}}$ locally Zariski-closed in $\mathcal{V}_o$ that

$$\mathrm{codim}_{\mathcal{V}_o} \mathcal{V}_o^{\mathrm{sm,nonreg}} \geq 2M + 1. \tag{2}$$

Let $\mathcal{P}_d^N$ denote the vector space of homogeneous polynomials of degree d in the affine coordinates $z_1, \ldots, z_N$, $N \leq M + k$, introduced above. Then

$$\mathcal{V}_o \cong \mathcal{P}_1^{M+k} \times \prod_{i=1}^{k} \prod_{j=1}^{d_i} \mathcal{P}_j^{M+k},$$

by identifying the dehomogenized sections in $H^0(\mathbb{P}^{M+k}, \mathcal{O}_{\mathbb{P}^{M+k}}(j))$ vanishing in o with $\mathcal{P}_j^{M+k}$ and decomposing the dehomogenisation $f_i = q_{i,1} + \cdots q_{i,d_i}$ of the F_i in the tuple $(L, F_1, \ldots, F_k) \in \mathcal{V}_o$ into homogeneous parts $q_{i,j}$ of degree j.

Consider the projection s of $\mathcal{V}_o$ to $\mathcal{P}_1^{M+k} \times \prod_{i=1}^{k} (\mathcal{P}_1^{M+k})$, that is, to the linear form l coming from L and the linear parts $q_{i,1}$ of the dehomogenized F_i. The map s is not defined everywhere on $\mathcal{V}_o$ but since $\mathcal{V}_o^{\mathrm{sm,nonreg}}$ consists of tuples $(L, F_1, \ldots, F_k)$ such that $V = \{F_1 = \ldots = F_k = 0\} \subset \mathbb{P}^{M+k}$ is smooth of codimension k in o and $l_{|T_oV} \not\equiv 0$, hence none of the l and $q_{i,1}$ can be 0, we conclude that s is defined on $\mathcal{V}_o^{\mathrm{sm,nonreg}}$.

Therefore, it is enough to show that the codimension of the isomorphic s-fibers in $\mathcal{V}_o^{\mathrm{sm,nonreg}}$ is $\geq 2M + 1$. Choosing the coordinates z_* such that $l = z_M, q_{i,1} = z_{M+1}, \ldots, q_{k,1} = z_{M+k}$, that means to show that the set $U(d_1, \ldots, d_k)$ of tuples of homogeneous polynomials in variables $z_1, \ldots, z_{M-1}$ defined by

$$\left\{ (q_{i,j})_{1 \leq i \leq k, 2 \leqslant j \leqslant d_i} \,|\, (q_{i,j})_{(i,j) \in I, j \neq 1} \text{ is not a regular sequence in } \mathcal{O}_{0,\mathbb{A}^{M-1}} \right\}$$

and Zariski-closed in $\prod_{i=1}^{k} \prod_{j=2}^{d_i} \mathcal{P}_j^{M-1}$ satisfies

$$\mathrm{codim}_{\prod_{i=1}^{k} \prod_{j=2}^{d_i} \mathcal{P}_j^{M-1}} U(d_1, \ldots, d_k) \geqslant 2M + 1. \tag{3}$$

In order to obtain this estimate only the degrees of the homogeneous polynomials $q_{i,j}$ matter. So we will discuss the codimension of the set of tuples

$$U := \{(q_i)_{1\leqslant i \leq M} \mid (q_i)_{1\leqslant i\leqslant M-2} \text{ is not a regular sequence in } \mathcal{O}_{0,\mathbb{A}^{M-1}}\}$$

in $\prod_{i=1}^{M} \mathcal{P}_{m_i}^{M-1}$ where $2 \leqslant m_1 \leqslant \cdots \leqslant m_M = d_k$ and we have

$$k_d := \#\{d_l : d_l \geqslant d, 1 \leqslant l \leqslant k\}$$

polynomials q_i of degree d. In particular $k = k_2 \geqslant k_3 \geqslant \cdots \geqslant k_{d_k}$ and

$$\sum_{i=1}^{M} m_i = \sum_{d=2}^{d_k} k_d \cdot d = \sum_{l=1}^{k}\sum_{d=2}^{d_l} d = \sum_{l=1}^{k} \frac{d_l(d_l+1)}{2} - k.$$

Let $Z(q_1,\ldots,q_j) \subset \mathbb{A}^{M-1}$ denote the vanishing locus of $q_1,\ldots,q_j$ in $\mathbb{A}^{M-1}$. Since the q_i are homogeneous $Z(q_1,\ldots,q_j)$ is a cone over the origin in $\mathbb{A}^{M-1}$, and we denote its projectivization in $\mathbb{P}^{M-2}$ by $V(q_1,\ldots,q_j)$. In particular, $(q_1,\ldots,q_j)$ is regular in $0 \in \mathbb{A}^{M-1}$ if and only if

$$\operatorname{codim}_{\mathbb{P}^{M-2}} V(q_1,\ldots,q_j) = \operatorname{codim}_{\mathbb{A}^{M-2}} Z(q_1,\ldots,q_j) = j,$$

where the codimension is set to be the minimum of the codimensions of the irreducible components.

Consequently, U is covered by locally Zariski-closed subsets $U_j \times \prod_{i=j+1}^{M} \mathcal{P}_{m_i}^{M-1}$, where $1 \leqslant j \leqslant M-2$ and

$$U_j := \{(q_1,\ldots,q_j) \mid \operatorname{codim}_{\mathbb{P}^{M-2}} V(q_1,\ldots,q_j) = \operatorname{codim}_{\mathbb{P}^{M-2}} V(q_1,\ldots,q_{j-1}) = j-1\}$$

is a locally Zariski-closed subset of $\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}$. Thus we need to show

$$\operatorname{codim}_{\prod_{i=1}^{k} \mathcal{P}_{m_i}^{M-1}} U = \min_{1\leqslant j\leqslant M-2} \operatorname{codim}_{\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}} U_j \geqslant 2M+1, \tag{4}$$

and this means to verify

$$\operatorname{codim}_{\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}} U_j \geqslant 2M+1 \text{ for } 1 \leqslant j \leqslant M-2. \tag{5}$$

If $m_j = 2$ (that is, $1 \leqslant j \leqslant k_2 = k$) then we estimate the codimension of U_j in $\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}$ using the method of [4]: Choose a general $(j-2)$-dimensional hyperplane $S \subset \mathbb{P}^{M-2}$. Then the projection $\pi_S : \mathbb{P}^{M-2} \to \mathbb{P}^{M-1-j}$ restricts to a finite morphism on each of the irreducible components of $V(q_1,\ldots,q_{j-1})$ for given $q_1,\ldots,q_{j-1}$ such that $\operatorname{codim}_{\mathbb{P}^{M-2}} V(q_1,\ldots,q_{j-1}) = j-1$. Consequently, polynomials in $\mathcal{P}_{m_j}^{M-1} = \mathcal{P}_2^{M-1}$ obtained as a pullback of a homogeneous quadratic polynomial defined on $\mathbb{P}^{M-1-j}$ do not vanish on $V(q_1,\ldots,q_{j-1})$. The linear space W_j of such pulled-back quadratic polynomials has dimension $\binom{M-1-j+2}{2}$. By construction, W_j intersects the space of polynomials in $\mathcal{P}_2^{M-1}$ vanishing on one of the irreducible components of $V(q_1,\ldots,q_{j-1})$ only in 0. Therefore,

$$\operatorname{codim}_{\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}} U_j \geqslant \binom{M-j+1}{2} \geqslant \binom{M-k+1}{2},\ j = 1,\ldots,k. \tag{6}$$

Note that this estimate also holds if $j=1$. Furthermore, $\binom{M-k+1}{2} \geqslant 2M+1$ since $3k+4 \leqslant M$. Hence (5) is shown for $j = 1,\ldots,k$.

If $m_j > 2$ (that is, $k+1 \leqslant j \leqslant M-2$) then we estimate the codimension of U_j in $\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}$ using the method of [5]: If $(q_1, \ldots, q_j) \in U_j$ choose an irreducible component B of $V(q_1, \ldots, q_j)$. By definition of U_j, the codimension of B in $\mathbb{P}^{M-2}$ is $j-1$. Assume that the codimension of the linear subspace $\langle B \rangle \subset \mathbb{P}^{M-2}$ spanned by B is b; that means in particular that $0 \leqslant b \leqslant j-1$.

Lemma 1. If $b < j-1$, then there are indices $1 \leqslant i_1 < \cdots < i_{j-1-b} \leqslant j-1$ such that B is an irreducible component of

$$V(q_{i_1}, \ldots, q_{i_{j-1-b}}) \cap \langle B \rangle.$$

Proof. Since $\mathrm{codim}_{\mathbb{P}^{M-2}} \langle B \rangle = b < j-1 = \mathrm{codim}_{\mathbb{P}^{M-2}} B$ one of the $q_1, \ldots, q_{j-1}$ must not vanish on $\langle B \rangle$. Let i_1 be the smallest index such that $q_{i_1 | \langle B \rangle} \not\equiv 0$. If $b = j-2$ one of the irreducible components of $V(q_{i_1}) \cap \langle B \rangle$ must be B and we are done. For $b < j-2$ choose an irreducible component R_1 of $V(q_{i_1}) \cap \langle B \rangle$ containing B. Since $\dim R_1 > \dim B$ and $q_{i|R_1} \equiv 0$ for $i = 1, \ldots, i_1$ we can find an index $i_1 + 1 \leqslant i_2 \leq j-1$ such that q_{i_2} does not vanish on R_1: Otherwise, $R_1 \subset V(q_1, \ldots, q_{j-1})$, and this is a contradiction to $\mathrm{codim}_{\mathbb{P}^{M-2}} V(q_1, \ldots, q_{j-1}) = j-1$. In the same way we can inductively find $i_3, \ldots, i_{j-1-b}$ such that finally B is an irreducible component of $V(q_{i_1}, \ldots, q_{i_{j-1-b}}) \cap \langle B \rangle$. Q.E.D. for the lemma.

The lemma implies that U_j is contained in the union of all locally Zariski-closed subsets $U_j(P; i_1, \ldots, i_{j-1-b}) \subset \prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}$ of the form

$$\left\{ (q_1, \ldots, q_j) \;\middle|\; \begin{array}{l} \text{there exists an irreducible component } B \text{ of } V(q_{i_1}, \ldots, q_{i_{j-1-b}}) \cap P : \\ \langle B \rangle = P, \ \mathrm{codim}_P B = j-1-b, \ q_{i|B} \equiv 0 \text{ for all } i = 1, \ldots, j \end{array} \right\},$$

where P ranges over all codimension b linear subspaces of $\mathbb{P}^{M-2}$, $0 \leqslant b \leqslant j-1$, and the indices $i_1, \ldots, i_{j-1-b}$ range over all increasing sequences $1 \leqslant i_1 < \cdots < i_{j-1-b} \leqslant j-1$. Note that for $b = j-1$ we just consider the sets

$$U_j(P) := \left\{ (q_1, \ldots, q_j) \;\middle|\; \begin{array}{l} P \text{ is an irreducible component} \\ \text{of } V(q_1, \ldots, q_{j-1}) \text{ and } q_{j|P} \equiv 0 \end{array} \right\} \subset \prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}.$$

Since the dimension of the Grassmann variety $\mathbb{G}(M-2-b, M-2)$ is $b(M-1-b)$, estimate (5) will follow if

$$\mathrm{codim}_{\prod_{i=1}^{j} \mathcal{P}_{m_i}^{M-1}} U_j(P; i_1, \ldots, i_{j-1-b}) \geqslant 2M + 1 + b(M-1-b) \tag{7}$$

holds. By a linear change of coordinates we can assume that

$$P = \{z_{M-b} = \cdots = z_{M-1}\}.$$

Let $q_{i_1}, \ldots, q_{i_{j-1-b}}$ be polynomials in $\prod_{r=1}^{j-1-b} \mathcal{P}_{m_{i_r}}^{M-1}$ such that an irreducible component B of $V(q_{i_1}, \ldots, q_{i_{j-1-b}})$ lies in P with $\mathrm{codim}_P B = j-1-b$ and $\langle B \rangle = P$. Then for any degree m, products of m linear forms

$$\prod_{i=1}^{m} (a_{i,1} z_1 + \cdots a_{i,M-b-1} z_{M-b-1})$$

do not vanish on B. These products span a linear subspace of $\mathcal{P}_m^{M-1}$ of dimension $(M-b-2)m + 1$ intersecting the linear subspace of polynomials vanishing on B

only in 0. We apply these facts to the polynomials $q_i \in \mathcal{P}^{M-1}_{m_i}$ with $i \in \{1,\dots,j\} - \{i_1,\dots,i_{j-1-b}\}$ and obtain

$$\begin{aligned}\operatorname{codim}_{\prod_{i=1}^{j}\mathcal{P}^{M-1}_{m_i}} U_j(P;i_1,\dots,i_{j-1-b}) &\geqslant \left(\sum_{\substack{i=1\\ i\notin\{i_1,\dots,i_{j-b-1}\}}}^{j} m_i\right)(M-b-2)+b+1\\ &\geqslant \left(\sum_{i=1}^{b} m_i + m_j\right)(M-b-2)+b+1.\end{aligned}$$

So (7) follows from

$$\left(\sum_{i=1}^{b} m_i + m_j - b\right)(M-b-2) \geq 2M. \tag{8}$$

Since $m_i \geqslant 2$ for $i=1,\dots,j-1$ and we consider the case $m_j \geqslant 3$ we have

$$\sum_{i=1}^{b} m_i + m_j - b \geqslant b+3.$$

The polynomial $(b+3)(M-2-b)-2M = -b^2+(M-5)b+M-6$ quadratic in b increases for $b \leqslant \frac{M-5}{2}$ and decreases for $b \geqslant \frac{M-5}{2}$. Since $M \geqslant 3k+4 \geqslant 7$, hence $3(M-2)-2M \geqslant 0$, estimate (8) is shown for $b=0, M-5$, hence for all $0 \leqslant b \leqslant M-5$. This leaves the cases $b=M-4, M-3$.

If $b=M-4$ then $j=M-3$ or $M-2$, and by the assumptions on the degrees m_i we have

$$\sum_{i=1}^{b} m_i + m_j \geqslant \sum_{i=1}^{M} m_i - 2d_k - (d_k-1) = \sum_{l=1}^{k-1}\frac{d_l(d_l+1))}{2} + \frac{(d_k-3)(d_k-2)}{2} + 2 - k.$$

Similarly, if $b=M-3$ then $j=M-2$, and

$$\sum_{i=1}^{b} m_i + m_j \geqslant \sum_{i=1}^{M} m_i - d_k - d_k = \sum_{l=1}^{k-1}\frac{d_l(d_l+1))}{2} + \frac{(d_k-2)(d_k-1)}{2} + 1 - k.$$

This last case is the worst possible situation: $V(q_1,\dots,q_{M-3})$ is a line in $\mathbb{P}^{M-2}$, and q_{M-2} vanishes on this line.

Solving an optimization problem and using $\sum_{l=1}^{k} d_l = M+k$ we obtain that

$$\sum_{l=1}^{k-1} d_l^2 + (d_k-3)^2 \geqslant k\left(\frac{M-3+k}{k}\right)^2 \quad\text{and}\quad \sum_{l=1}^{k-1} d_l^2 + (d_k-2)^2 \geqslant k\left(\frac{M-2+k}{k}\right)^2.$$

Consequently, (8) follows for $b=M-4, M-3$ if

$$\left[\frac{(M-3+k)^2}{2k} + \frac{M-3+k}{2} - k - M + 6\right]\cdot 2 \geqslant 2M \tag{9}$$

and

$$\left[\frac{(M-2+k)^2}{2k} + \frac{M-2+k}{2} - k - M + 3\right]\cdot 1 \geqslant 2M. \tag{10}$$

Now (9) is equivalent to

$$k \leqslant \frac{(M-3)^2}{M} \tag{11}$$

and (10) is equivalent to

$$k \leqslant \frac{(M-2)^2}{3M-2}. \tag{12}$$

Both inequalities are satisfied if $M \geqslant 3k+4$. Then $k \geqslant 1$ implies $M \geqslant 7$, so no further lower bound on M is needed. Proof of Theorem 7 is complete.

5. Hypertangent divisors. Let us prove Theorem 5. The argument is similar to the proof of Theorem 4 in [7], so we will only sketch the main steps.

Step 1. Assume that the pair $(V, \frac{1}{n}D)$ is not canonical for an effective divisor $D \sim nH$, where $H \in \operatorname{Pic} V$ is the class of a hyperplane section, $K_V = -H$. By linearity of all conditions involved in this assumption, the divisor D can be assumed to be irreducible. For some prime divisor E over V the inequality $\operatorname{ord}_E D > na(E, V)$ holds.

We look at the centre $B \subset V$ of the divisor E. Since by [7, Proposition 3.6] for any irreducible subvariety $Y \subset V$ of dimension at least k (where $k = \operatorname{codim}(V \subset \mathbb{P})$) we have $\operatorname{mult}_Y D \leqslant n$, we conclude that $\dim B \leqslant k-1$.Let $o \in B$ be a point of general position, $\sigma\colon V^+ \to V$ its blow up and $E^+ = \sigma^{-1}(o)$ the exceptional divisor. A standard argument (see, for example, [7, Proposition 2.5]) gives us a hyperplane $\Delta \subset E^+ \cong \mathbb{P}^{M-1}$ satisfying the inequality

$$\operatorname{mult}_o D + \operatorname{mult}_\Delta D^+ > 2n,$$

where D^+ is the strict transform of D on V^+.

Now let T be a general hyperplane section of V, containing the point o and cutting out Δ on E^+: that is to say, $T^+ \cap E^+ = \Delta$. It is easy to see that the effective cycle of the scheme-theoretic intersection $D_T = (D \circ T)$ is well defined and satisfies the estimate

$$\operatorname{mult}_o D_T > 2n. \tag{13}$$

We will consider D_T as an effective divisor on the complete intersection $T \subset \mathbb{P}^{M+k-1}$ of codimension k, $D_T \sim nH_T$.

Step 2. By Sec. 3-4, the complete intersection $T \subset \mathbb{P}^{M+k-1}$ satisfies the regularity condition. Namely, for a system of linear coordinates $z_1, \dots, z_{M+k-1}$ with the origin at the point o, the variety T is given by a system of non-homogeneous polynomial equations:

$$\begin{array}{rcl}
\bar{f}_1 & = & \bar{q}_{1,1} + \cdots + \bar{q}_{1,d_1}, \\
 & \cdots & \\
\bar{f}_i & = & \bar{q}_{i,1} + \cdots + \bar{q}_{i,d_i}, \\
 & \cdots & \\
\bar{f}_k & = & \bar{q}_{k,1} + \cdots + \bar{q}_{k,d_k},
\end{array}$$

where the bar means the restriction onto the hyperplane $\{l = 0\}$ — the hyperplane that cuts out T on V. Now the set of homogeneous polynomials

$$\{\bar{q}_{i,j} \mid (i,j) \in I\}$$

forms a regular sequence in $\mathcal{O}_{o,\mathbb{P}^{M+k-1}}$.

Step 3. Now we can apply the technique of hypertangent divisors [9, Chapter 3] to the divisor D_T on the complete intersection $T \subset \mathbb{P}^{M+k-1}$ in precisely the same way as it was done in [7, Section 3] or, in more details, in [8, Section 5] and obtain the estimate

$$\frac{\mathrm{mult}_o D_T}{2n} \leqslant \max\left\{1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_+}{d_+ - 1}\right\},$$

where $d_+ = d_k$, if $d_{k-1} = d_k$, and $d_+ = d_k - 1$, otherwise. It is easy to see that if $d_k \geqslant 8$, this gives us the inequality $\mathrm{mult}_o D_T \leqslant 2n$, which contradicts the estimate (13) obtained in Step 1. The contradiction completes the proof of Theorem 5.

Proof of Theorem 6 follows the same lines and repeats the argument given in [8, Section 4] word for word. What is different from the proof of Theorem 5 given above, is the starting point: assuming the inequality

$$\mathrm{ct}(V) \leqslant \frac{M}{M+1},$$

we obtain for any rational number $\lambda > \frac{M}{M+1}$ an effective divisor $D \sim nH$ such that the pair $(V, \frac{\lambda}{n} D)$ is not canonical. Now, repeating the proof of Theorem 5, we use the technique of hypertangent divisors to obtain the inequality

$$1 < \lambda \max\left\{1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_+}{d_+ - 1}\right\},$$

and taking the limit as $\lambda \to \frac{M}{M+1}$, we conclude that

$$\max\left\{1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_+}{d_+ - 1}\right\} \geqslant \frac{M+1}{M}.$$

However, it is a trivial check that in the assumptions of Theorem 6 the last inequality can not be true. Q.E.D. for Theorem 6.

References

[1] Cheltsov I.A. and Shramov K.A., Log canonical thresholds of non-singular Fano three-folds. Russian Math. Surveys. **63** (2008), no. 5, 73-180.

[2] Demailly J.-P. and Kollár J., Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann. Sci. de l'École Norm. Sup. **34** (2001), 525-556.

[3] Nadel A., Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. Ann. Math. **132** (1990), 549-596.

[4] Pukhlikov A. V., Birational automorphisms of Fano hypersurfaces, Invent. Math. **134** (1998), no. 2, 401-426.

[5] Pukhlikov A. V., Birationally rigid Fano complete intersections, Crelle J. für die reine und angew. Math. **541** (2001), 55-79.

[6] Pukhlikov A. V., Birational geometry of Fano direct products, Izvestiya: Mathematics, **69** (2005), no. 6, 1225-1255.

[7] Pukhlikov A. V. Birational geometry of algebraic varieties with a pencil of Fano complete intersections, Manuscripta Mathematica. **121** (2006), 491-526.

[8] Pukhlikov A. V., Existence of the Kähler-Einstein metric on certain Fano complete intersections, Math. Notes **88**, No. 4, 33-39.

[9] Pukhlikov Aleksandr, Birationally Rigid Varieties. Mathematical Surveys and Monographs **190**, AMS, 2013.

[10] Tian G., On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$. Invent. Math. **89** (1987), 225-246.

Thomas Eckl and Aleksandr Pukhlikov, Department of Mathematical Sciences, The University of Liverpool, Mathematical Sciences Building, Liverpool, L69 7ZL, England, U.K.

E-mail address: eckl@liv.ac.uk, pukh@liv.ac.uk